

# Two-way and three-way negativities of three qubit entangled states

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In this letter we propose to quantify three qubit entanglement using global negativity along with  $K$ -way negativities, where  $K = 2$  and  $3$ . The principle underlying the definition of  $K$ -way negativity for pure and mixed states of  $N$ -subsystems is PPT sufficient condition. However,  $K$ -way partial transpose with respect to a subsystem is defined so as to shift the focus to  $K$ -way coherences instead of  $K$  subsystems of the composite system.

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Quantum entanglement is not only a fascinating aspect of multipartite quantum systems, but also a physical resource needed for quantum communication, quantum computation and information processing in general. Bipartite entanglement is well understood, however, many aspects of multipartite entanglement are still to be investigated. Peres [1] and the Horodecki [2, 3, 4] have shown a positive partial transpose (PPT) of a bipartite density operator to be a sufficient criterion for classifying bipartite entanglement. Negativity [5, 6] based on Peres Horodecki criterion has been shown to be an entanglement monotone [7, 8, 9]. Negativity is a useful concept being related to the eigenvalues of partially transposed state operator and can be calculated easily. In this letter, we define 2-way and 3-way negativities and propose a classification of three qubit states based on measures related to global, 2-way and 3-way negativities. General definition of  $K$ -way negativities for pure and mixed states of  $N$ -subsystems is given in ref [10]. The  $K$ -way partial transpose with respect to a subsystem is defined so as to shift the focus to  $K$ -way coherences instead of  $K$  subsystems of the composite system. While pure  $K$ -partite entanglement of a composite system is generated by  $K$ -way coherences,  $K$ -partite entanglement can, in general, be present due to  $(K - 1)$  way coherences as well.

## I. GLOBAL NEGATIVITY

The Hilbert space,  $C^d = C^{d_1} \otimes C^{d_2} \otimes C^{d_3}$ , associated with a quantum system composed of three qubits is spanned by basis vectors  $|i_1 i_2 i_3\rangle$ , where  $i_m = 0$  or  $1$ , and  $m = 1, 2, 3$ . Here  $d_m = 2$  is the dimension of Hilbert space associated with  $m^{th}$  qubit. To simplify the notation we denote the vector  $|i_1 i_2 i_3\rangle$  by  $\left| \prod_{m=1}^3 i_m \right\rangle$  and write a general three qubit pure state as

$$\hat{\rho} = \sum_{\substack{i_1, i_2, i_3, \\ j_1, j_2, j_3}} \left\langle \prod_{m=1}^3 i_m \right| \hat{\rho} \left| \prod_{m=1}^3 j_m \right\rangle \left| \prod_{m=1}^3 i_m \right\rangle \left\langle \prod_{m=1}^3 j_m \right|. \quad (1)$$

To measure the overall entanglement of a subsystem  $p$ , we shall use twice the negativity as defined by Vidal and Werner [6], and call it global Negativity  $N_G^p$ . It is an entanglement measure based on Peres-Horodecki [1, 2] NPT (Negative partial transpose) sufficient criterion for classifying bipartite entanglement. The global partial transpose of  $\hat{\rho}$  (Eq.(1)) with respect to sub-system  $p$  is defined as

$$\hat{\rho}_G^{T_p} = \sum_{\substack{i_1, i_2, i_3, \\ j_1, j_2, j_3}} \left\langle j_p \prod_{m=1, m \neq p}^3 i_m \right| \hat{\rho} \left| i_p \prod_{m=1, m \neq p}^3 j_m \right\rangle \left| \prod_{m=1}^3 i_m \right\rangle \left\langle \prod_{m=1}^3 j_m \right|. \quad (2)$$

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The partial transpose  $\hat{\rho}_G^{T_p}$  of an entangled state is not positive. Global Negativity, defined as

$$N_G^p = \left\| \rho_G^{T_p} \right\|_1 - 1, \quad (3)$$

is an entanglement monotone and measures the entanglement of subsystem  $p$  with its complement in a bipartite split of the composite system. Global negativity vanishes on ppt-states and is equal to the entropy of entanglement on maximally entangled states.

## II. $K$ -WAY PARTIAL TRANSPOSE

The  $K$ -way partial transpose of  $\rho$  with respect to subsystem  $p$  is obtained by transposing the indices of subsystem  $p$  in those matrix elements,  $\left\langle \prod_{m=1}^3 i_m \left| \hat{\rho} \right| \prod_{m=1}^3 j_m \right\rangle$ , that satisfy the condition  $\sum_{m=1}^3 |j_m - i_m| = K$ , where  $K = 0$  to  $3$ .

A typical matrix element of the three qubit state operator  $\hat{\rho}$  involves a change of state of  $K$  subsystems, where  $K = 0$  to  $3$ . For example, a matrix element involving a change of state of two subsystems looks like  $\langle i_1 i_2 i_3 | \hat{\rho} | j_1 j_2 i_3 \rangle$  ( $i_1 \neq j_1, i_2 \neq j_2$ ). The set of two distinguishable subsystems that change state while one of the sub-systems does not, can be chosen in three distinct ways. In general, the number of spins that are flipped to get a vector  $|j_1 j_2 j_3\rangle$  from the vector  $|i_1 i_2 i_3\rangle$  is  $K = \sum_{m=1}^3 |j_m - i_m|$ . The operator  $\hat{\rho}$  can be split up into parts labelled by  $K$  ( $0 \leq K \leq 3$ ) and written as

$$\hat{\rho} = \sum_{K=0}^3 \hat{R}_K, \quad (4)$$

with

$$\hat{R}_K = \sum_{I_K} \left\langle \prod_{m=1}^3 i_m \left| \hat{\rho} \right| \prod_{m=1}^3 j_m \right\rangle \left| \prod_{m=1}^3 i_m \right\rangle \left\langle \prod_{m=1}^3 j_m \right|. \quad (5)$$

Here  $I_K = \{i_1, i_2, i_3, j_1, j_2, j_3 : (\sum_m |j_m - i_m|) = K\}$ . The 2-way and 3-way partial transpose with respect to qubit  $p$  are defined as

$$\begin{aligned} \hat{\rho}_2^{T_p} &= \sum_{K=0,1,3} \hat{R}_K + \\ &\sum_{I_2} \left\langle j_p \prod_{\substack{m=1 \\ m \neq p}}^3 i_m \left| \hat{\rho} \right| i_p \prod_{\substack{m=1 \\ m \neq p}}^3 j_m \right\rangle \left| \prod_{m=1}^3 i_m \right\rangle \left\langle \prod_{m=1}^3 j_m \right| \end{aligned} \quad (6)$$

and

$$\begin{aligned} \hat{\rho}_3^{T_p} &= \sum_{K=0}^2 \hat{R}_K + \\ &\sum_{I_3} \left\langle j_p \prod_{\substack{m=1 \\ m \neq p}}^3 i_m \left| \hat{\rho} \right| i_p \prod_{\substack{m=1 \\ m \neq p}}^3 j_m \right\rangle \left| \prod_{m=1}^3 i_m \right\rangle \left\langle \prod_{m=1}^3 j_m \right|. \end{aligned} \quad (7)$$

## III. $K$ -WAY NEGATIVITY

The  $K$ -way negativity calculated from  $K$ -way partial transpose of matrix  $\rho$  with respect to subsystem  $p$ , is defined as

$$N_K^p = \left\| \rho_K^{T_p} \right\|_1 - 1, \quad (8)$$

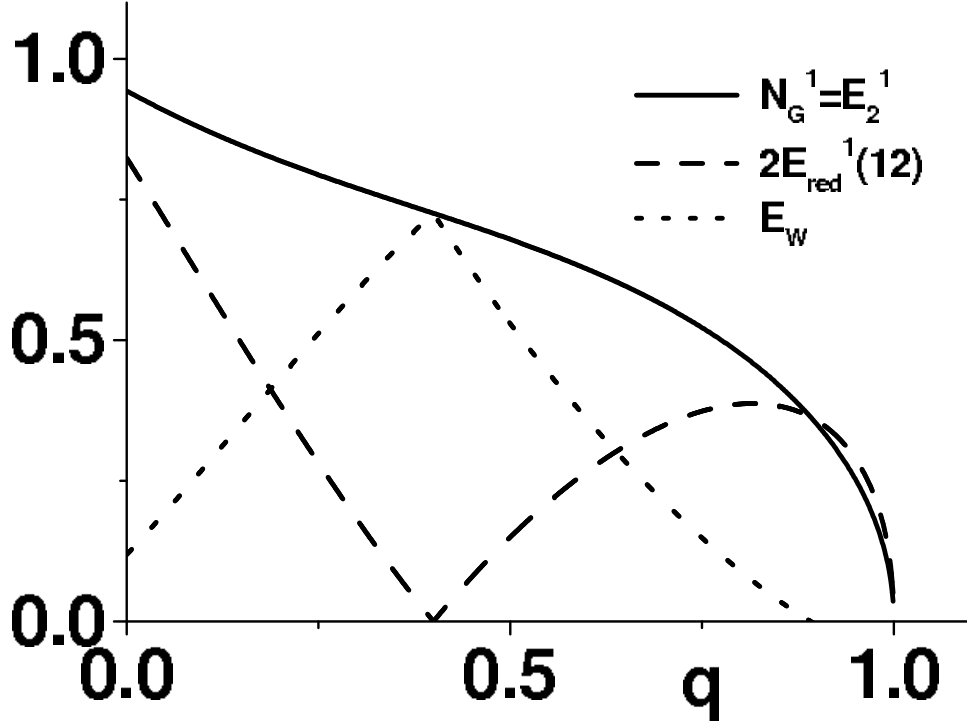


FIG. 1:  $N_G^1$ ,  $2E_{red}^1(12)$  and  $E_W$  versus parameter  $q$  for the state  $\hat{\rho}_1$ .

where  $\|\rho_K^{T_p}\|_1$  is the trace norm of  $\rho_K^{T_p}$ . Using the definition of trace norm and the fact that  $\text{tr}(\rho_K^{T_p}) = 1$ , we get  $\|\rho_K^{T_p}\|_1 = 2 \sum_i |\lambda_i^{K-}| + 1$ ,  $\lambda_i^{K-}$  being the negative eigenvalues of matrix  $\rho_K^{T_p}$ . The negativity,  $N_K^p = 2 \sum_i |\lambda_i^{K-}|$  ( $p = 1, 2, 3$ ), depends on  $K$ -way coherences and is a measure of all possible types of entanglement attributed to  $K$ -way coherences. Intuitively, for a system to have pure  $N$ -partite entanglement, it is necessary that  $N$ -way coherences are non-zero. On the other hand,  $N$ -partite entanglement can be generated by  $(N-1)$ -way coherences, as well. For a three qubit system, maximally entangled tripartite GHZ state is an example of pure tripartite entanglement involving 3-way coherences. The global negativity  $N_G^p = N_3^p = 1$ , for maximally entangled three qubit GHZ state. Maximally entangled W-state is a manifestation of tripartite entanglement due to 2-way coherences. For pure states as well as those mixed states for which the density matrix is positive, entanglement of a subsystem is completely determined by global Negativity  $N_G^p$  and the hierarchy of negativities  $N_K^p$  ( $K = 2, \dots, N$ ), calculated from  $\rho_K^{T_p}$  associated with the  $p^{th}$  sub-system. For three qubit system,  $N_2^p$ ,  $N_3^p$ , and  $N_G^p$  ( $p = 1, 2, 3$ ) quantify the coherences present in the composite system.

#### IV. HOW MUCH BI AND TRIPARTITE ENTANGLEMENT IS GENERATED BY 2-WAY AND 3-WAY NEGATIVITIES?

A natural question is, how much of global negativity comes from 2-way transpose and how much has its origin in 3-way transpose, for a given qubit? The operator  $\hat{\rho}_G^T$  in its eigen basis is written as

$$\hat{\rho}_G^T = \sum_i \lambda_i^{G+} |\Psi_i^{G+}\rangle \langle \Psi_i^{G+}| + \sum_i \lambda_i^{G-} |\Psi_i^{G-}\rangle \langle \Psi_i^{G-}|, \quad (9)$$

where  $\lambda_i^{G+}$  and  $|\Psi_i^{G+}\rangle$  ( $\lambda_i^{G-}$  and  $|\Psi_i^{G-}\rangle$ ) are the positive (negative) eigenvalues and eigenvectors, respectively. As such the negativity of  $\hat{\rho}_G^T$  is given by

$$N_G^p = -2 \sum_i \langle \Psi_i^{G-} | \hat{\rho}_G^T | \Psi_i^{G-} \rangle = 2 \sum_i |\lambda_i^{G-}|. \quad (10)$$

It is easily verified that

$$\hat{\rho}_G^T = \hat{\rho}_2^T + \hat{\rho}_3^T - \hat{\rho}. \quad (11)$$

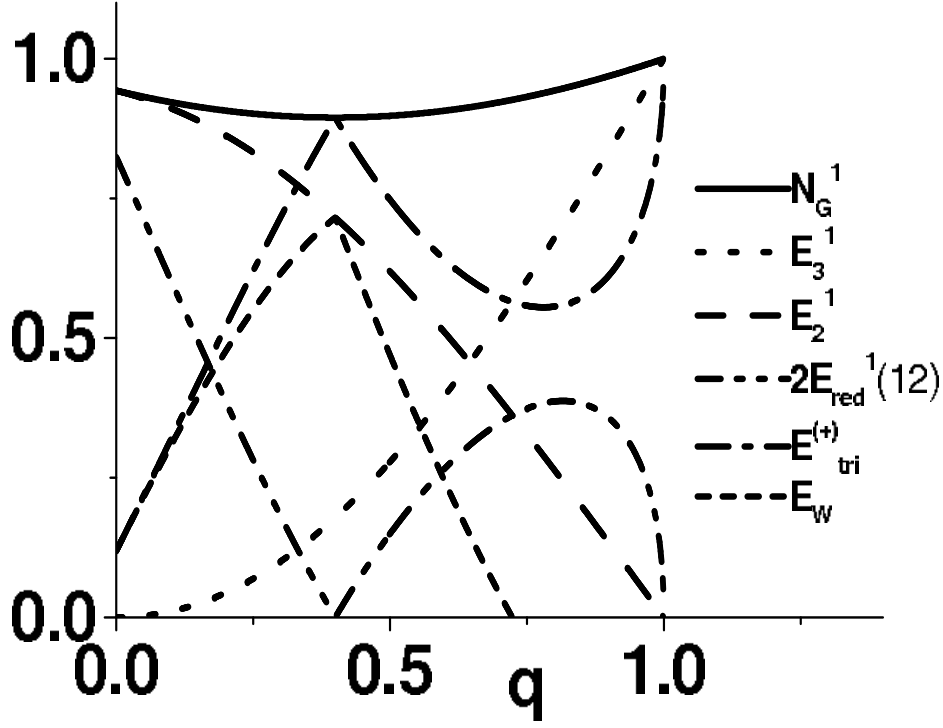


FIG. 2:  $N_G^1$ ,  $E_3^1$ ,  $E_2^1$ ,  $2E_{red}^1(12)$ ,  $E_{tri}^{(+)}$  and  $E_W$  as a function of parameter  $q$  for the state  $\Psi_2^{(+)}$ .

Substituting Eq. (11) in Eq. (10) and recalling that  $\hat{\rho}$  is a positive operator with trace one, we get

$$\begin{aligned} N_G^p &= -2 \sum_i \langle \Psi_i^{G-} | \hat{\rho}_2^{T_p} | \Psi_i^{G-} \rangle - 2 \sum_i \langle \Psi_i^{G-} | \hat{\rho}_3^{T_p} | \Psi_i^{G-} \rangle, \\ &= E_2^p + E_3^p, \end{aligned} \quad (12)$$

where  $E_K^p = -2 \sum_i \langle \Psi_i^{G-} | \hat{\rho}_K^{T_p} | \Psi_i^{G-} \rangle$  is the contribution of  $K$ -way partial transpose to  $N_G^p$ .

The set of states that can be transformed into each other by local unitary operations lie on the same orbit and have the same entanglement as the canonical state expressed in terms of the minimum number of independent vectors [11, 12]. Construction of pure three qubit canonical state has been given by Acin et al [13, 14]. It is easily verified that for the states reducible to the canonical state by local unitary operations, although  $N_G^p$  is invariant under local operations,  $\sum_{p=1}^3 N_K^p$  varies under local unitary operations. For a canonical state,  $N_3 = \sum_{p=1}^3 N_3^p$ , is found to lie at a minimum with respect to local unitary rotations applied to any of the three qubits. We conjecture that for a canonical state,  $\min(E_3^1, E_3^2, E_3^3)$  is a measure of genuine 3-way entanglement of three qubit system.

Bipartite entanglement of qubit one with qubit two equals the negativity  $E_{red}^1(12)$  of  $\hat{\rho}_{red}^{T_1}(12)$  where the reduced operator,  $\hat{\rho}_{red}(12) = \text{tr}_3(\hat{\rho})$ . In case no W-like tripartite entanglement is present, the bipartite entanglement of a qubit is given by  $E_{red}^1(12) + E_{red}^1(13)$ . For a canonical state, the measure  $E_2^p$  contains information about the W-like as well as pairwise entanglement of qubit  $p$ .

## V. THREE QUBIT GHZ AND W-TYPE STATES

Three qubit Greenberger-Horne-Zeilinger state

$$\Psi_{GHZ} = \frac{1}{\sqrt{2}} (|000\rangle + |111\rangle), \quad \hat{\rho}_{GHZ} = |\Psi_{GHZ}\rangle \langle \Psi_{GHZ}| \quad (13)$$

is a maximally entangled state having genuine tripartite entanglement. For this state  $E_2^p = 0$ , and  $E_3^p = N_G^p = 1.0$ , for  $p = 1, 2, 3$ . On the other hand there exists a class of tripartite states akin to maximally entangled W-state given

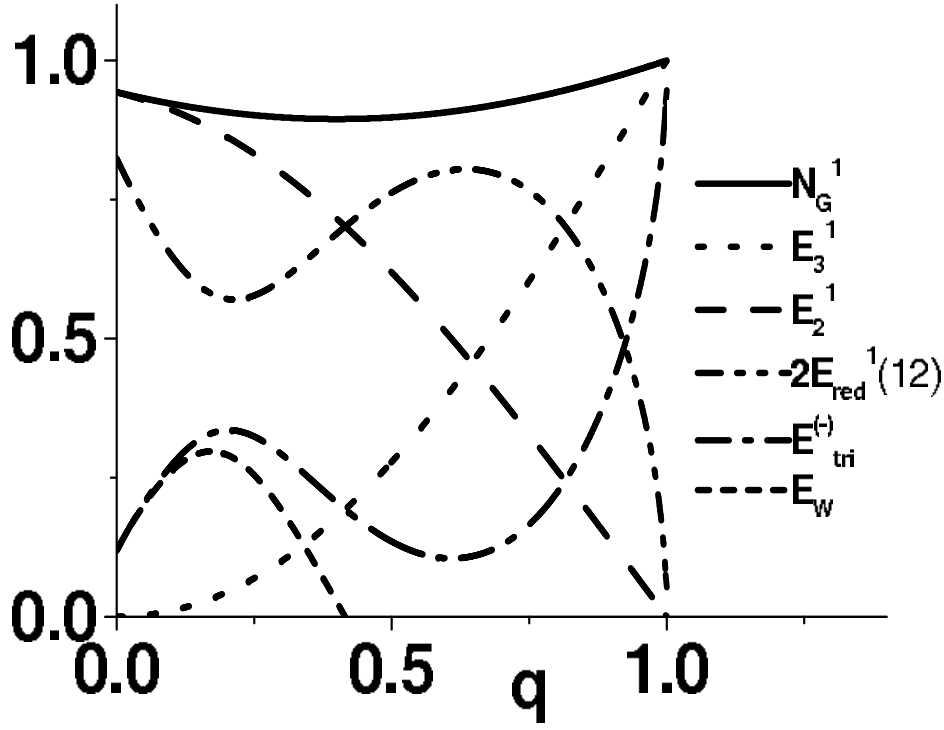


FIG. 3:  $N_G^1$ ,  $E_3^1$ ,  $E_2^1$ ,  $2E_{red}^1(12)$ ,  $E_{tri}^{(-)}$  and  $E_W$  as a function of parameter  $q$  for the state  $\Psi_2^{(-)}$ .

by

$$\Psi_W = \frac{1}{\sqrt{3}} (|100\rangle + |010\rangle + |001\rangle), \quad \hat{\rho}_W = |\Psi_W\rangle \langle \Psi_W|. \quad (14)$$

For the pure state  $\Psi_W$ ,  $E_2^p = N_G^p = 0.94$ , for  $p = 1, 2, 3$ . Bipartite entanglement of qubit one and two in the state  $\Psi_W$  is measured by the negativity of partially transposed reduced density operator  $\hat{\rho}_{red}(12)$  ( $\hat{\rho}_{red}(12) = tr_3(\hat{\rho}_W)$ ), which is found to be  $E_{red}^1(12) = 0.41$ . The total pairwise entanglement of a qubit in W-state is twice the value of  $E_{red}^p(12)$  ( $=0.82$ ) and is less than  $E_2^p (=0.94)$ . The residue accounts for the W-type tripartite entanglement of the system and generates  $E_W = N_G^p - 2E_{red}^p(12) = 0.12$ .

The three qubit state

$$\begin{aligned} \hat{\rho}_1 = & \frac{q}{2} |111\rangle \langle 111| \\ & + \left| \sqrt{\frac{q}{2}} |000\rangle + \sqrt{(1-q)} \Psi_W \right\rangle \left\langle \sqrt{\frac{q}{2}} |000\rangle + \sqrt{(1-q)} \Psi_W \right| \end{aligned} \quad (15)$$

has no genuine tripartite entanglement and no 3-way coherences as such  $N_G^1 = N_2^1$ . Analogous to the case of state  $\Psi_W$ , we have  $E_{red}(12) = 2E_{red}^1(12)$  and  $E_W = N_G^1 - 2E_{red}^1(12)$ . Fig. 1 displays  $N_G^1$ ,  $2E_{red}^1(12)$  and  $E_W$  as a function of parameter  $q$ .

To decipher the interplay of genuine tripartite entanglement and entanglement generated by 2-way coherences, we examine the coherences of single parameter pure states

$$\Psi_2^{(+)} = \sqrt{q} \Psi_{GHZ} + \sqrt{(1-q)} \Psi_W, \quad \hat{\rho}_2^+ = \left| \Psi_2^{(+)} \right\rangle \left\langle \Psi_2^{(+)} \right| \quad 0 \leq q \leq 1 \quad (16)$$

and

$$\Psi_2^{(-)} = \sqrt{q} \Psi_{GHZ} - \sqrt{(1-q)} \Psi_W, \quad \hat{\rho}_2^- = \left| \Psi_2^{(-)} \right\rangle \left\langle \Psi_2^{(-)} \right| \quad 0 \leq q \leq 1. \quad (17)$$

For a given value of  $q$ , the state may have bipartite, genuine tripartite as well as W-type entanglement as seen by  $N_G^1$ ,  $E_3^1$ , and  $E_2^1$  displayed in Figs. (2) and (3). Bipartite entanglement ( $E_{red}(12) = 2E_{red}^1(12)$ ) of a qubit in a

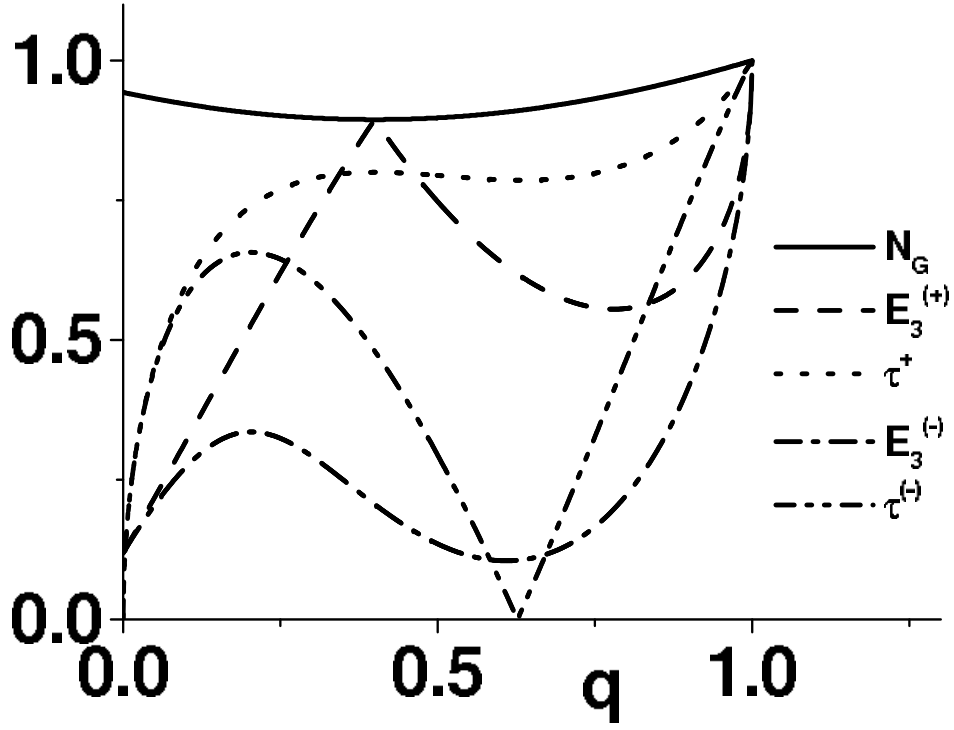


FIG. 4:  $N_G^1$ ,  $E_3^+$ ,  $\tau^+$ ,  $E_3^-$ , and  $\tau^-$  as a function of parameter  $q$  for the states  $\Psi_2^{(+)}$  and  $\Psi_2^{(-)}$ .

state  $\hat{\rho}_2 = |\Psi_2\rangle\langle\Psi_2|$  is obtained by calculating the negativity  $E_{red}^1(12)$  of partially transposed  $\hat{\rho}_{red}(12)$  defined as  $\hat{\rho}_{red}(12) = Tr_3(\hat{\rho}_2)$ . We may remark that the states  $\hat{\rho}_1$  and  $\hat{\rho}_2^+$  share the same  $\hat{\rho}_{red}(12)$  with  $E_{red}^1(12)$  going to zero at  $q = 0.4$ . Figs. (2) and (3) also display total bipartite entanglement of a qubit,  $2E_{red}^1(12)$ , total tripartite entanglement  $E_{tri}^{(\pm)} = (N_G^1)^{\pm} - 2E_{red}^{1\pm}(12)$  and W-state like entanglement  $E_W^1 = N_G^1 - E_3^1 - 2E_{red}^1(12)$ , of a qubit in the states  $\Psi_2^{(+)}$  and  $\Psi_2^{(-)}$ , respectively. Tripartite entanglement  $E_{tri}^{(\pm)}$ , resulting from the interference of  $N_W^1$  and  $N_3^1$  shows that 2-way coherence is partially annihilated by the 3-way coherence and vice versa. Wootters three tangle calculated from concurrence [15, 16] has been used to study tripartite entanglement of three qubit system in ref. [17]. Three tangle for states  $\Psi_2^{(\pm)}$ , calculated from

$$\tau^{(\pm)} = \left| q^2 \pm \frac{8\sqrt{6(q(1-q))^3}}{9} \right|,$$

and tripartite entanglement  $E_{tri}^{(\pm)}$  are plotted as a function of  $q$  in Fig. 4. The two measures show similar trend and match at  $q = 1$ , but in general 3-tangle overestimates the tripartite entanglement of the three qubit composite system in comparison with  $E_{tri}^{(\pm)}$ .

In summary, we have extended the use of PPT criteria to the case of tripartite entanglement by splitting the partially transposed three qubit state operator into matrices that contain information either about bipartite or about genuine tripartite entanglement. The set of 2-way, 3-way and global negativities make clear distinction between the coherences and nature of entanglement present in three qubit composite states. The three qubit canonical states may be classified using  $N_G^p$ ,  $E_2^p$ ,  $E_3^p$ ,  $E_{red}^p(pq)$  and  $E_W^p$  ( $p, q = 1$  to  $3$ ,  $p \neq q$ ) as labels. These entanglement measures are easy to calculate using standard diagonalization routines. We believe that the relations between  $K$ -way negativities and entanglement of a composite quantum system, a) with sub-system dimensions higher than two, and b) with more than three subsystems, can be figured out. We expect our results to help advance the investigation of multipartite entanglement.

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